Mathathon 2019
Round 2

Maths and Physics Club, IIT Bombay

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1. Suppose that π could be written in the form $\dfrac{p}{q}$, where p and q are natural numbers.

   Define the family of integrals I_n for $n = 0, 1, 2, \cdots$ by

   $$I_n = \frac{q^{2n}}{n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2 \right)^n \cos x dx$$

   (a) By appropriate manipulation of the integral, deduce that

   $$I_n = (4n-2)q^2 I_{n-1} - p^2 q^2 I_{n-2}, \text{ for } n \geq 2.$$

   (b) Prove that $\dfrac{p}{q} \left(\dfrac{p}{2} \right)^{2n} \dfrac{1}{n!} < 1$, if n is sufficiently large.
   Deduce that π is irrational.

2. Let $x_1, x_2, \cdots, x_n$ (where $n \in \mathbb{N}$) be the solutions to the following system of linear equations:

   $$\begin{aligned}
   \frac{x_1}{2^2 - 1^2} + \frac{x_2}{2^2 - 3^2} + \cdots + \frac{x_n}{2^2 - (2n-1)^2} &= 1 \\
   \frac{x_1}{4^2 - 1^2} + \frac{x_2}{4^2 - 3^2} + \cdots + \frac{x_n}{4^2 - (2n-1)^2} &= 1 \\
   &\vdots \\
   \frac{x_1}{4n^2 - 1^2} + \frac{x_2}{4n^2 - 3^2} + \cdots + \frac{x_n}{4n^2 - (2n-1)^2} &= 1
   \end{aligned}$$

   Let S_n denote $\displaystyle\sum_{k=1}^{n} x_k$.
   Show that S_n is a perfect square for infinitely many values of n.

   You may use the following fact:
   Given $D \in \mathbb{N}$ such that D is not a perfect square, there exist infinitely many $(x, y) \in \mathbb{Z}^2$ such that:

   $$x^2 - Dy^2 = 1$$

3. Let $x,\ y,\ n$ be positive integers with $n > 1$. How many ordered triples $(x,\ y,\ n)$ of solutions are there to the equation $x^n - y^n = 2^{100}$?

4. Let $x_1, x_2, \cdots, x_{2k+1}$ be $2k+1$ variables which are randomly chosen with uniform distribution in $(0,1)$. $(k \in \mathbb{N} \cup \{0\})$

$$N := \sum_{i=1}^{2k+1} \left\lfloor \frac{1}{x_i} \right\rfloor$$

Let $P(k)$ denote the probability that N is odd. Hence, evaluate:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \big(2P(k) - 1\big)$$

5. Consider solutions to the equation

$$x^2 - cx + 1 = \frac{f(x)}{g(x)},$$

where f and g are polynomials with nonnegative real coefficients. For each $c > 0$, determine the minimum possible degree of f, or show that no such f, g exist.
(Note that by solutions, we mean a pair of functions f and g such that the equation given holds for all $x \in \mathbb{R}$.)

6. Let x, y, and z be real numbers such that $x^4 + y^4 + z^4 + xyz = 4$.
Show that

$$x \le 2 \text{ and } \sqrt{2 - x} \ge \frac{y+z}{2}.$$

7. Find all polynomial functions $P(x)$ with real coefficients that satisfy

$$P(x\sqrt{2}) = P(x + \sqrt{1 - x^2})$$

for all real x with $|x| \le 1$.

8. The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are written on the faces of a regular octahedron so that each face contains a different number. Find the probability that no two consecutive numbers are written on faces that share an edge, where 8 and 1 are considered consecutive.